Morita Equivalence - RW Sec 3.2

Two C^*-algebras A, B are Morita equivalent if $\exists$ an A-B imprimitivity bi-module $_AX_B$
(IBM)

Recall $_AX_B$ is an IBM if

ⓐ X is a full left Hilbert A-mod, full right Hilbert B-mod
$\underbrace{\text{left Hilbert } A\text{-mod}}_{\operatorname{span}\{_A\langle x,y\rangle\} = A}$

ⓑ $\forall\ x,y \in X,\ a \in A,\ b \in B$

$$\langle a\cdot x, y\rangle_B = \langle x, a^*\cdot y\rangle_B \qquad {}_A\langle x\cdot b, y\rangle = {}_A\langle x, y\cdot b^*\rangle$$

ⓒ $\forall\ x, y, z \in X$

$${}_A\langle x,y\rangle\cdot z = x\cdot\langle y,z\rangle_B$$

<u>Thm</u> Morita equiv. is an equiv. rel.

Pf: <u>reflexive</u>

Every C^*-algebra A is an A-A IBM ✓

$${}_A\langle a,b\rangle = ab^*, \quad \langle a,b\rangle_A = a^*b$$

<u>Symmetric</u>

Given $_AX_B$, Need $_BY_A$

Define the <u>dual module</u>:

$\overline{X} = \{\overline{x} \mid x \in X\}$ is the conjugate vector space of X.

$$\overline{x} + \overline{y} = \overline{x+y}$$
$$\lambda\overline{x} = \overline{\overline{\lambda}x}$$

$\overline{X}$ is a B-A bi-mod with

$$b\cdot\overline{x} = \overline{x\cdot b^*} \qquad \overline{x}\cdot a = \overline{a^*\cdot x}$$

$${}_B\langle\overline{x},\overline{y}\rangle = \langle x,y\rangle_B \qquad \langle\overline{x},\overline{y}\rangle_A = {}_A\langle x,y\rangle$$

NTS $\bar{X}$ is a B-A IBM

Since ${}_A X_B$ is an A-B IBM

ⓐ X is a full $\underbrace{\text{left Hilbert } A\text{-mod}}_{\overline{\text{span}}\{{}_A\langle x,y\rangle\} = A}$, full right Hilbert B-mod

ⓑ $\forall\ x,y \in X,\ a \in A,\ b \in B$

$$\langle a\cdot x, y\rangle_B = \langle x, a^*\cdot y\rangle_B \qquad {}_A\langle x\cdot b, y\rangle = {}_A\langle x, b^*\cdot y\rangle$$

ⓒ $\forall\ x,y,z \in X$

$${}_A\langle x,y\rangle\cdot z = x\cdot\langle y,z\rangle_B$$

ⓐ $\overline{\text{span}}\{\langle \bar{x},\bar{y}\rangle_A\} = \overline{\text{span}}\{{}_A\langle x,y\rangle\} = A$

ⓑ $\forall\ \bar{x},\bar{y} \in \bar{X},\ b \in B$

$${}_B\langle b\cdot\bar{x},\bar{y}\rangle = {}_B\langle \overline{x\cdot b^*},\bar{y}\rangle = \langle x\cdot b^*, y\rangle_B = \langle x, y\cdot b\rangle_B = {}_B\langle \bar{x}, b^*\cdot\bar{y}\rangle$$

ⓒ $\forall\ \bar{x},\bar{y},\bar{z} \in \bar{X}$

$$\bar{x}\cdot\langle\bar{y},\bar{z}\rangle_A = \overline{{}_A\langle y,z\rangle^*\cdot x} = \overline{{}_A\langle z,y\rangle\cdot x} = \overline{z\cdot{}_A\langle y,x\rangle} = {}_B\langle\bar{z},\bar{y}\rangle\cdot\bar{x}$$

So ${}_B\bar{X}_A$ is an IBM ✓

<u>Transitive</u>

Given ${}_A X_B$, ${}_B Y_C$

Construct the internal tensor product

$$Z = X \otimes_B Y = \frac{X \odot Y}{(x\cdot b)\otimes y = x \otimes (b\cdot y)}$$

Z is an A-C bi-mod with

actions: $a\cdot(x\otimes y) = (a\cdot x)\otimes y$

$(x\otimes y)\cdot c = x\otimes(y\cdot c)$

inner prods: ${}_A\langle\langle x_1\otimes y_1, x_2\otimes y_2\rangle\rangle = {}_A\langle x_1, x_2\cdot{}_B\langle y_2, y_1\rangle\rangle$

$$\langle\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle\rangle_C = \langle \langle x_2, x_1 \rangle_B \cdot y_1, y_2 \rangle_C$$

From Shen (Prop 3.11):

$${}_A\|z\| := \|{}_A\langle\langle z,z\rangle\rangle\|^{1/2} = \|\langle\langle z,z\rangle\rangle_C\|^{1/2} = \|z\|_C$$

complete Z wrt this norm, get an A-C Hilbert bi-mod

check imprimitivity:

ⓐ $\text{span}\{{}_A\langle\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle\rangle\} = \text{span}\{{}_A\langle x_1, x_2 \cdot \underbrace{{}_B\langle y_2, y_1 \rangle}_{\text{dense in } B} \rangle\}$ is dense in A

<u>Fact</u>: If X_B is a Hilbert B-mod, then $X \cdot B$ is dense in X.

ⓑ easy

ⓒ good exercise ✓

Thus Morita equivalence is an equivalence relation ∎

isomorphic $\Rightarrow$ Morita equiv. $\Rightarrow$ same K-theory

<u>Examples</u>

From Shen: $\underbrace{{}_{K(\mathcal{H})}\mathcal{H}_{\mathbb{C}} \; ; \; {}_{M_n(A)}A^n_A}$; ${}_{C_0(G/H) \rtimes G}C^*(G)_{C^*(H)}$

G locally compact group

$H \leq G$, closed

$K(\mathcal{H}) \sim M_n(\mathbb{C}) \sim \mathbb{C}$ $\qquad {}_{M_n(\mathbb{C})}\mathbb{C}^n_{\mathbb{C}}$

$$\Downarrow$$

$$M_n(\mathbb{C}) \sim M_k(\mathbb{C})$$

Also, for T a LCH space, $\mathcal{H}$ is a Hilbert space

$${}_{C_0(T, K(\mathcal{H}))}C_0(T, \mathcal{H})_{C_0(T)}$$

An Equivalent characterization

Def Let $p \in M(A)$ be a projection ($p = p^* = p^2$).

The C^*-subalgebra $pAp = \{pap \mid a \in A\}$ is called a corner

eg: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$

$p \in M_2(\mathbb{C})$

Def Corners pAp, qAq are complementary if $p + q = 1_{M(A)}$

pAp is full if $\overline{ApA} := \overline{\operatorname{span}}\{apb \mid a, b \in A\} = A$

From then: ${}_{\overline{ApA}}Ap_{pAp}$ is an IBM ($Ap = \{ap \mid a \in A\}$)

If pAp is full then $\overline{ApA} = A \Rightarrow {}_A Ap_{pAp}$

If pAp, qAq are complementary & full then

${}_A Ap_{pAp}$, ${}_A Aq_{qAq} \Rightarrow pAp \sim qAq$

Thm C^*-algebras A, B are Morita equiv.

$\iff$

$\exists$ a C^*-alg C with complementary full corners st

$pCp \cong A$, $qCq \cong B$

Pf: ($\Leftarrow$)

$A \sim pCp \sim qCq \sim B$

($\Rightarrow$) Let ${}_A X_B$ be an IBM, ${}_B \tilde{X}_A$ it's dual module

Construct C, the linking algebra of ${}_A X_B$, as follows:

Let $M = X \oplus B$, for $x, y \in X$, $a \in A$, $b \in B$, define

$L = \begin{bmatrix} a & x \\ \tilde{y} & b \end{bmatrix} : M \to M$ by $\begin{bmatrix} a & x \\ \tilde{y} & b \end{bmatrix}\begin{bmatrix} z \\ c \end{bmatrix} = \begin{bmatrix} a \cdot z + x \cdot c \\ \langle y, z \rangle_B + bc \end{bmatrix}$

Then $L^* = \overline{L}^T = \begin{bmatrix} a^* & y \\ \overline{x} & b^* \end{bmatrix}$ so $L \in \mathcal{L}(M)$

$C := \left\{ \begin{bmatrix} a & x \\ \overline{y} & b \end{bmatrix} \,\middle|\, x,y \in X,\ a \in A,\ b \in B \right\}$ is a $*$-subalg of $\mathcal{L}(M)$

with usual matrix addition and scalar mult, and

$$\begin{bmatrix} a_1 & x_1 \\ \overline{y_1} & b_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 & x_2 \\ \overline{y_2} & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + {}_A\langle x_1, y_2\rangle & a_1 \cdot x_2 + x_1 \cdot b_2 \\ \overline{y_1} \cdot a_2 + b_1 \cdot \overline{y_2} & \langle y_1, x_2\rangle_B + b_1 b_2 \end{bmatrix}$$

C is norm-closed by Lemma 3.20

$\max\{\|a\|, \|x\|_B, \|y\|_B, \|b\|\} \leq \|L\|_{op} \leq \|a\| + \|x\|_B + \|y\|_B + \|b\|$ $(*)$

Need complementary full corners iso to A, B

$A \hookrightarrow C$ via $a \longmapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$

$B \hookrightarrow C$ via $b \longmapsto \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$

This is injective by $(*)$

View A, B as C^*-subalgebras of C

Then $p = \begin{bmatrix} 1_{M(A)} & 0 \\ 0 & 0 \end{bmatrix}$, $q = \begin{bmatrix} 0 & 0 \\ 0 & 1_{M(B)} \end{bmatrix}$ these are projections in $M(C)$, $p + q = 1_{M(C)}$

giving $pCp \cong A$, $qCq \cong B$

NTS complementary

ie $CpC = \left\{ \begin{bmatrix} a_1 a_2 & a_1 \cdot x \\ \overline{y} \cdot a_2 & \langle y, x\rangle_B \end{bmatrix} \right\}$ and $CqC = \left\{ \begin{bmatrix} {}_A\langle x, y\rangle & x \cdot b_2 \\ b_1 \cdot \overline{y} & b_1 b_2 \end{bmatrix} \right\}$

have dense spans in C.

For CqC: Fix $L = \begin{bmatrix} a & x \\ \overline{y} & b \end{bmatrix}$ in C

Since ${}_AX$ is full: $\overline{\text{span}}\{{}_A\langle x,y\rangle\} = A$

So approx a by $\sum {}_A\langle x_i, y_i\rangle \longrightarrow \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$

<u>Prop 2.31</u> X_B a $HC^*M \Rightarrow \forall\, x \in X\ \exists!\ x_0 \in X$ st $x = x_0 \cdot \langle x_0, x_0\rangle_B$

So write $x = x_0 \cdot \langle x_0, x_0\rangle_B$

$\overline{y} = \overline{y_0 \cdot \langle y_0, y_0\rangle_B} = {}_B\langle \overline{y_0}, \overline{y_0}\rangle \cdot \overline{y_0}$

Write $b = b_1 b_2$

(can do $bu_\lambda \to b$ for some approximate unit (u_λ))

add all this together to get an approx of L

by elts of $\text{span}\{C_q C\}$

$\blacksquare$

<u>cor</u> $C = K(M)$ where $M = X \oplus B$ as a Hilb B-mod